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GPS/IRS AIME: Calculation of Thresholds and Protection Radius Using Chi-Square Methods

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Ref 3

BIOGRAPHY

Dr. John Diesel is presently a scientist at Litton Aero Products. He holds several patents, and has published several technical papers on navigation and control systems. He received the PhD. in E. E. and Mathematics from Washington University in St. Louis in 1959, and is a member of Sigma Xi.

Ms. Sherry Luu is a senior engineer at Litton Aero Products. She is currently working on IRS/GPS integration, and has been responsible for much of the C language programming in Litton's latest IRS software mechanizations. She received a B. S. in Computer Science from California State University at Northridge.

ABSTRACT

A mathematical justification for AIME is developed, and it is compared with the mathematical justification for RAIM. It is seen that the mathematical bases for the two approaches is essentially the same. The test statistic for AIME is based on the residuals in the innovations process of the Kalman filter, rather than the residuals from the instantaneous "snapshot" least squares solution used in RAIM. This is logical since the Kalman filter residual is the difference between measured Pseudo Range to each satellite and the predicted Pseudo Range from the estimated solution, which is the least squares solution based on all past measurements.

Rather than the parity transformation used in RAIM, AIME transforms the residuals to the principal axes (eigenvectors) of the ellipsoid for the n dimensional normal distribution of the residuals. It is shown that any deterministic failure of a satellite leads to a non-central chi square distribution. This makes it possible to determine the exact probabilities for failure detection and exclusion on a single covariance run, which corresponds to an infinite number of Monte Carl runs.

INTRODUCTION

Autonomous Integrity Monitored Extrapolation (AIME) is a software mechanization for integrating GPS with IRS to solve the GPS integrity/availability problem (References 1, 2). AIME can be used to achieve primary means of navigation for en-route, terminal area, and non-precision approach, without WAAS.

Using Kalman filter principles, AIME generates a least-squares solution based on the entire history of GPS measurements. There are 24 parameters estimated in each of the many parallel Kalman filters used in AIME, in addition to four parameters estimated separately. These include all of the significant user clock bias states, baro error states, and inertial error states. Also included are DC bias error states for each satellite. This means that once the integity of a new satellite is established, only changes in the DC bias over 2.5 minute averaging intervals need be monitored to detect satellite failures.

At the request of RTCA Special Committee 159, Working Group 3A (GPS/INS), a mathematical basis for the AIME failure detection and exclusion (FDE) algorithm was developed. It is comparable to that used for the RAIM FDE, as developed by R. Grover Brown in Reference 3. The mathematical basis for RAIM is first reviewed in this paper. The corresponding mathematical basis for AIME is then presented, with corresponding equation numbers so that they can be compared step by step.

In RAIM, the instantaneous "snapshot" least squares residual vector is used to compute the test statistic. In AIME, the Kalman filter innovations process residual vector is used to compute the test statistic. In both cases, the test statistic is chi-square distributed, which is the basis for computing the failure detection and exclusion (FDE) probabilities.

RAIM FDE

(This review is a summary of the methods described by R. Grover Brown in Reference 3).

1. RAIM uses n satellite measurements y (nx1) to obtain a least squares estimate of four parameters x (4x1).

The four estimated parameters x are the errors in the three position coordinates, and the aircraft receiver clock bias, which have been obtained from previous estimates. The measurements y are the differences between measured pseudoranges and predicted ones, based on the previous estimates. The linearized relationship between y and x is:

$$y = G \cdot x + \epsilon \qquad (R1)$$

where G (nx4) is a linear observation matrix, depending on satellite geometry. Its first three columns are the direction cosines to each of the satellites, and a column of ones corresponding to the receiver clock bias error. The measurement error ϵ (nx1) may have both random and deterministic parts.

Assuming $n > 4$, the least squares estimate is obtained as:

$$x_{LS} = (G^T G)^{-1} G^T y \qquad (R2)$$

2. RAIM determines integrity from the range residual vector w (nx1).

The range residual vector is defined by first determining the measurement change y_{LS} (nx1) which results from the least squares fit solution:

$$y_{LS} = G \cdot x_{LS} \qquad (R3)$$

Th range residual vector w (nx1) is then the measurement difference:

$$w = y - y_{LS} = y - G \cdot x_{LS} \qquad (R4)$$

3. Satellite failures are detected by using the magnitude of the parity vector p (n-4)x(1) as the test statistic.

The range residual vector is first transformed by the parity transformation P ((n-4)x(n)):

$$p = P \cdot w \qquad (R5)$$

The sum-squared residuals are then given by:

$$p^2 = p^T \cdot p = w^T \cdot w \qquad (R6)$$

where p is the magnitude of the parity vector.

If there are no satellite failures, it is assumed that the random part of the measurement error ϵ is normally distributed with zero mean. The sum-squared residual p^2 (1x1) is chi-square distributed with n-4 degrees of freedom (DOF). The threshold p_D for detecting failures is selected from this distribution to result in the false alarm rate $P_{FA} = 10^{-5}$/hour, assuming no satellite failures.

4. The effect of a deterministic range bias error b in the ith satellite is determined.

The parity vector resulting from this range bias error is given by:

$$dp_i = P \cdot b_i \qquad (R7)$$

where vector b_i (nx1) has a b in row i, and zero elements elsewhere. The magnitude of this parity vector is denoted dp_i.

From Equation (R2) for the least squares solution, the effect of this same error b_i on the solution is given by:

$$dx_i = (G^T G)^{-1} G^T \cdot b_i \qquad (R8)$$

The horizontal (radial) position error is obtained from the first two components, dx_{i1}, and dx_{i2}, of dx_i. The horizontal error dR_i is:

$$dR_i = (dx_{i1}^2 + dx_{i2}^2)^{1/2} \qquad (R9)$$

The linear relationship between the horizontal error and the test statistic is given by the characteristic slope:

$$\mathrm{SLOPE}(i) = dR_i / dp_i \qquad (R10)$$

In this relationship, the assumed magnitude b of the original range bias error has cancelled out.

5. The test statistic p^2 resulting from both the random noise with zero mean and the deterministic bias error b is next considered.

This statistic has a non-central chi square distribution. When the corresponding horizontal error exceeds a specified protection level HPL, it is desired that this statistic exceed the threshold p_D so that the satellite failure is detected.

The critical bias p_{bias} is defined as the deterministic value of the test statistic which exceeds th detection threshold by a sufficient margin that the probability

P_{miss} of not exceeding the threshold is less than 0.001, when the random noise is added. The probability P_{miss} is also called the probability of missed detection.

6. The horizontal protection level (HPL) is determined.

The satellite whose failure is most difficult to detect is the one with the smallest test statistic when the horizontal limit exceeds the protection level. This is the satellite with the maximum slope, $SLOPE_{max}$, as computed in Equation (R10). The HPL is then computed as:

$$HPL = SLOPE_{max} \cdot p_{bias} \qquad \text{(R11)}$$

The value of p_{bias} *is determined* from the non-central chi-square distribution with n-4 degrees of freedom. It is the non-centrality parameter which results in a probability less than P_{miss} of p being less than the detection threshold p_D.

7. If a fault is detected, using 3 above, the faulty satellite is excluded.

This is done by selecting the jth subset of n-1 satellites with the smallest normalized test statistic. (Reference 4, Appendix K). However, for navigation to continue, the test statistic must be less than a 99.9% decision threshold. Also, the HPL_j based on this threshold must be less than the horizontal alert limit (HAL) for that phase of flight.

Th reason the decision threshold can be temporarily reduced to the 99.9% level is that the false alarm rate is determined by the original fault detection threshold in 3 above. Once a fault is detected, the probability of failed exclusion is less than 0.001, using the reduced threshold, regardless of which satellite is faulty.

AIME FDE

Both the step numbers and the equation numbers in the following correspond to the respective numbers for RAIM above. By comparing the two approaches, it can be seen that the mathematical justification for them is essentially the same. In both cases, the probabilities for detection and exclusion are based on both the central and the non-central chi square distributions.

1. AIME uses (nxt) satellite measurements z(t) (nxt) to obtain a least squares estimate of 24 parameters x (24x1)

The 24 estimated parameters are the error states x(t) in a Kalman filter. The measurements z(t) are the differences, at 1 Hz., between the measured pseudoranges and predicted ones, based on the previous estimates. These measurements are pre-filtered by averaging over the Kalman filter cycle time $(t_k - t_{k-1})$ = 150 seconds (2.5 minutes). The linearized relationship between the averaged measurements z and the residual errors x is:

$$\mathbf{z}(k) = \mathbf{H}(k) \cdot \mathbf{x}(k) + \mathbf{v}(k) \qquad \text{(A1)}$$

where H(k) (nx24) is an averaged observation matrix. The averaged measurement error v(k) (nx1) may have both random and deterministic parts.

For any number of satellites n > 0, the least squares estimate is obtained as:

$$\mathbf{x}^{+}(k) = \mathbf{x}^{-}(k) + \mathbf{K}(k) \cdot$$

$$[\mathbf{z}(k) - \mathbf{H}(k) \cdot \mathbf{x}^{-}(k)] \qquad \text{(A2)}$$

where $x^{-}(k)$ *is the previous least square estimate.*

2. AIME determines integrity from the residuals r(k) (nx1) in the Kalman filter innovations process.

If $x^{-}(k)$ is the estimate of the error state before the updates at cycle k, the predicted measurement is

$$\mathbf{z}^{-}(k) = \mathbf{H}(k) \cdot \mathbf{x}^{-}(k) \qquad \text{(A3)}$$

The Kalman filter residual (denoted Greek letter "nu" in the literature) is:

$$\mathbf{r}(k) = \mathbf{z}(k) - \mathbf{z}^{-}(k)$$

$$= \mathbf{z}(k) - \mathbf{H}(k) \cdot \mathbf{x}^{-}(k) \qquad \text{(A4)}$$

The components of r(k) have an n dimensional normal distribution with zero mean and known covariance:

$$E[\mathbf{r}(k)] = 0$$

$$E[\mathbf{r}(k) \cdot \mathbf{r}^{T}(k)] = \mathbf{V}(k)$$

where the covariance is:

$$\mathbf{V}(k) = \mathbf{H}(k) \cdot \mathbf{P}^{-}(k) \cdot \mathbf{H}^{T}(k) + \mathbf{R}(k)$$

If there are no satellite failures and the Kalman filter model is correct, the residual vectors are independent for different k (the "innovations property", Reference 5). The innovations property can be expressed as:

$$E[\mathbf{r}(j) \cdot \mathbf{r}^{T}(k)] = 0, \text{ if } j \text{ n.e. } k$$

3. *Satellite failures are detected by using the magnitude of the normalized residual vector* s (nx1) *as the test statistic.*

Since the covariance matrix V is symmetric and positive definite, its eigenvectors l_j (nx1) are orthogonal, and its eigenvalues $d_j = \sigma_j^2$ are positive (Reference 6). Denoting the modal matrix by **L** (nxn), and the diagonal matrix of eigenvalues by **D** (nxn), the residuals are transformed by:

$$\mathbf{r} = \mathbf{L} \cdot \mathbf{q}$$

$$\mathbf{q} = \mathbf{D}^{1/2} \cdot \mathbf{s} \qquad \text{(A5)}$$

The test statistic is the normalized sum square of the normalized transformed residual s (nx1). In Appendix A it is shown that:

$$s^2 = \mathbf{s}^T \cdot \mathbf{s} = \mathbf{r}^T \cdot \mathbf{V}^{-1} \cdot \mathbf{r} \qquad \text{(A6)}$$

If there are no satellite failures, it is shown in Appendix A that s^2 is chi square distributed with n degrees of freedom (DOF). The threshold s_D for detecting failures is selected to result in the false alarm rate $P_{FA} = 10^{-5}$/hour, assuming no satellite failures.

The innovations property makes it possible to detect very slow satellite drifts. This is done by estimating the mean of the residuals over a long time interval, to determine the averaged residual. To avoid contaminating the Kalman filter, both the measurements and residuals are stored in buffers for periods of 30 minutes or more.

Batch processing is used to determine the averages. Using the alternative form of the Kalman filter, the inverse covariance of the final estimate is obtained first:

$$\mathbf{V}^{-1}_{avg} = \underbrace{\Sigma_k \mathbf{V}^{-1}(k)}_{\text{sum of all } V^{-1}(k)}$$

The estimate of the mean is then obtained as:

$$\mathbf{r}_{avg} = (\mathbf{V}^{-1}_{avg})^{-1} \cdot \underbrace{\Sigma_k \mathbf{V}^{-1}(k) \cdot \mathbf{r}(k)}_{\text{sum of all } V^{-1}(k) \cdot r(k)}$$

The normalized sum-squared estimated mean residual is:

$$s^2_{avg} = (\mathbf{r}^T_{avg}) \cdot (\mathbf{V}^{-1}_{avg}) \cdot (\mathbf{r}_{avg})$$

As shown in Appendix A, this statistic has a central chi-square distribution if there is no failure. As shown in Appendix B, the distribution is non-central chi-square if there is a failure. By taking a long averaging interval, the covariance of the averaged residual becomes small. If there is a failure, the averaged residual does not become small, so that the statistic becomes large. This makes it possible to detect very slow satellite failures.

4. The effect of a range bias error b in the ith satellite is determined.

The transformed residual ds_i (nx1) resulting from this range bias error is given by:

$$ds_i = \mathbf{D}^{-1/2} \cdot \mathbf{L}^T \cdot \mathbf{b}_i \qquad \text{(A7)}$$

where vector $\mathbf{b}_i$ (nx1) has a b in row i, and zero elements elsewhere. The magnitude of this transformed residual vector is denoted ds_i.

From equation (A2), the effect of this same error on the solution is given by:

$$dx_i^+(k) = \mathbf{K}(k) \cdot \mathbf{b}_i \qquad \text{(A8)}$$

The horizontal (radial) position error is obtained from the first two components of dx_i. The horizontal error is :

$$dR_i = (dx_{i1}^2 + dx_{i2}^2)^{1/2} \qquad \text{(A9)}$$

The linear relationship between the horizontal error and the test statistic is given by the characteristic slope:

$$\text{SLOPE}(i) = dR_i / ds_i \qquad \text{(A10)}$$

In this relationship, the assumed magnitude b of the original range bias error has cancelled out.

5. The test statistic s^2 resulting from both the random noise with zero mean and the deterministic bias error b is next considered.

It is shown in Appendix B that this statistic has a non-central chi square distribution. The critical bias ds = s_{bias} is defined as the deterministic value of the test statistic which exceeds the detection threshold by a sufficient margin that the probability P_{miss} of the total statistic not exceeding the threshold is less than 0.001, when random noise is added.

6. The horizontal protection level is determined.

The satellite whose failure is most difficult to detect is the satellitte with maximum slope, SLOPE_{max}, as computed in Equation (A10). The HPL is:

$$\text{HPL} = \text{SLOPE}_{max} \cdot s_{bias} \qquad \text{(A11)}$$

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The value of s_{bias} is determined from the non-central chi-square distribution to give a probability less than P_{miss} of s being less than the detection threshold s_D.

7. If a fault is detected, using 3 above, the faulty satellite is excluded.

This is done by running a bank of n test Kalman filters in parallel with the least squares Kalman filter. Each of these test Kalman filters excludes a different satellite and uses a subset of n-1 satellites. If a failure is detected by the least squares filter, which uses all n satellites, the test Kalman filter with the smallest test statistic s_j is assumed to exclude the bad satellite.

For navigation to continue, this test statistic must be less than the 99.9% decision threshold. Also, the HPL_j for this threshold must be less than the HAL for that phase of flight.

CONCLUSIONS

Unlike RAIM, which only uses present measurements in a "snapshot" approach, AIME uses all present and past measurements to determine a least squares solution in a Kalman filter. This means that failure detection and exclusion does not require a minimum number of satellites in view. As a result, AIME can be used for primary navigation for all flight phases with a high availability.

To isolate and exclude failures, AIME uses parallel Kalman filters, each of which excludes different satellites. An additional filter is used to exclude baro or inertial failure modes. If a failure is detected using all satellites in view, the failure is isolated by comparing averaged residuals from each of the parallel Kalman filters.

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APPENDIX A. WHY THE TRANSFORMED TEST STATISTIC IS CHI SQUARE DISTRIBUTED

The modal transformation L transforms the residual vector r to a transformed residual q, whose components q_i are uncorrelated:

$$r = L \cdot q \qquad \text{(AA1)}$$

The covariance matrix V is transformed to diagonal form:

$$D = L^{-1} \cdot V \cdot L \qquad \text{(AA2)}$$

where

$$d_{ij} = 0, \text{ if } i \text{ n.e. } j,$$

$$d_{ii} = \sigma_i^2 \qquad \text{(AA3)}$$

Since V is symmetic, the eigenvectors are orthogonal. The modal matrix L is then an orthogonal transformation, so that $L^{-1} = L^T$. This transformation rotates the axes to the eigenvectors, which are the principal axes of the ellipsoid representing the n dimensional normal distribution.

The inverse V^{-1} is obtained from (AA2):

$$D^{-1} = L^{-1} \cdot V^{-1} \cdot L = L^T \cdot V^{-1} \cdot L$$

$$V^{-1} = L \cdot D^{-1} \cdot L^{-1} = L \cdot D^{-1} \cdot L^T \qquad \text{(AA4)}$$

The transformation Equation (AA1) and Equation (AA4) are then substituted into the definition of the transformed test statistic:

$$s^2 = r^T \cdot V^{-1} \cdot r$$

The substitution gives:

$$s^2 = q^T \cdot L^T \cdot L \cdot D^{-1} \cdot L^T \cdot L \cdot q$$

$$= q^T \cdot D^{-1} \cdot q$$

$$= \Sigma_i \, q_i^2/\sigma_i^2$$

$$= \Sigma_i \, s_i^2$$

Since the q_i are independent with variance σ_i^2, the s_i are independent, and are N(0,1). By definition, s^2 is chi square distributed with n DOF.

APPENDIX B. WHY THE TRANSFORMED STATISTIC WITH SATELLITE FAILURE IS NON-CENTRAL CHI SQUARE

If there is a failure in satellite i, resulting in range bias error b_i, the corresponding residual is denoted $dr = b_i$. More generally, dr could represent any satellite failure, such as a ramp. Also, the components of dr could be correlated if the failure extended over more than one Kalman filter cycle.

Since the Kalman filter is a time-varying linear system, the total residual due to a failure is determined by superposition as:

$$r_F = r + dr$$

where r is the residual with no satellite failure, and dr is the system response due to the failure alone.

The total residual with the satellite failure included is transformed into the total test statistic by:

$$s_F = D^{-1/2} \cdot L^T \cdot (r + dr)$$

$$= s + ds$$

where

$$s = D^{-1/2} \cdot L^T \cdot r$$

$$ds = D^{-1/2} \cdot L^T \cdot dr$$

Since s has an n dimensional spherical normal distribution, the axes can be rotated so that ds lies along one of the axes. This proves that

$$s_F^2 = s_F^T \cdot s_F$$

has a non-central chi square distribution with non-centrality parameter ds^2.